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MATHEMATICAL MODEL OF A HETEROGENEOUS MEDIUM CONSISTING OF A MATRIX AND SPHERICAL INCLUSIONS

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A mathematical model of a heterogeneous medium consisting of an elastoplastic matrix and elastic spherical inclusions is constructed. The model takes into account plastic zones, which appear in the vicinity of the inclusions. It is shown that when the effect of plastic zones is taken into account, the averaged "moduli" of volume compression, shear, and yield strength depend not only on the volume concentrations of the inclusions, but also on the mean pressure in the medium.

We consider a heterogeneous material consisting of a matrix with spherical inclusions of radius a. We assume that the volume concentration of the inclusions is small ($m_1 \ll 1$) and their yield strength is rather large. In this case, the inclusions do not contact each other directly and experience elastic deformation. This situation is typical of heterogeneous materials that have a metal matrix and inclusions in the form of carbide particles. The equation of state for the matrix and inclusions is assumed to be known; it is necessary to find an equation of state for the whole heterogeneous material, which is a composite. This equation of state should determine the relationship between the mean stresses and mean strains in the composite. Many papers are devoted to the solution of this problem; a list of these papers can be found in the monographs [1–3]. Nevertheless, the authors are not aware of any papers where the effect of the plastic zone appearing in the vicinity of an inclusion was discussed in deriving the equation of state.

We use the averaging technique developed previously in [4–6] for a porous elastoplastic material.

The heterogeneous material is divided into spherical cells of radius b with a spherical inclusion of radius a in the center of each cell (Fig. 1). The cell radius b is determined from the formula

$$b = a/m_1^{1/3}, \quad m_1 = (4/3)\pi a^3 n,$$

where n is the concentration of inclusions (the number of inclusions per unit volume) and m_1 is the volume concentration of inclusions (the fraction of unit volume occupied by inclusions). Averaging the actual stresses σ'_{ij} and strains ε'_{ij} over the cell volume, we obtain

$$\sigma_{ij} = m_1\sigma_{ij}^{(1)} + m_2\sigma_{ij}^{(2)}, \quad \varepsilon_{ij} = m_1\varepsilon_{ij}^{(1)} + m_2\varepsilon_{ij}^{(2)}, \tag{1}$$

where

$$\sigma_{ij} = \frac{1}{V}\int\limits_V \sigma'_{ij}\,dV, \quad \varepsilon_{ij} = \frac{1}{V}\int\limits_V \varepsilon'_{ij}\,dV, \quad \sigma_{ij}^{(k)} = \frac{1}{V_k}\int\limits_{V_k} \sigma'_{ij}\,dV, \quad \varepsilon_{ij}^{(k)} = \frac{1}{V_k}\int\limits_{V_k} \varepsilon'_{ij}\,dV,$$

$$m_k = \frac{V_k}{V}, \quad m_1 + m_2 = 1, \quad V_1 = \frac{4}{3}\pi a^3, \quad V_2 = \frac{4}{3}\pi(b^3 - a^3), \quad V = \frac{4}{3}\pi b^3,$$

$$\varepsilon_{ij} = \frac{1}{3}\varepsilon_{kk}\delta_{ij} + e_{ij}, \quad \sigma_{ij} = -p\delta_{ij} + S_{ij}, \quad i = 1,\,2,\,3, \quad j = 1,\,2,\,3,$$

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σ_{ij}, ε_{ij}, S_{ij}, and e_{ij} are the tensors of mean stresses, mean strains, and their deviators, and the superscripts (1) and (2) indicate the inclusion and the matrix, respectively. Assuming the strains to be small, we suppose that the material of inclusions is described by the Hooke's law

$$\varepsilon_{kk}^{(1)\prime} = -p^{(1)\prime}/K_1, \quad e_{ij}^{(1)\prime} = S_{ij}^{(1)\prime}/(2\mu_1), \tag{2}$$

where K_1 is the modulus of volume compression and μ_1 is the shear modulus; summation is performed over repeated indices. The matrix material obeys the Hooke's law in the elastic region and to the Prandtl–Reiss model in the plastic region:

$$\begin{gathered} \varepsilon_{kk}^{(2)\prime} = -p^{(2)\prime}/K_2, \\ e_{ij}^{(2)\prime} = \frac{S_{ij}^{(2)\prime}}{2\mu_2} \quad \text{for} \quad I_2^{(2)\prime} < Y_2^2, \quad I_2^{(2)\prime} = \frac{3}{2} S_{ij}^{(2)\prime} S_{ij}^{(2)\prime}, \\ \dot{e}_{ij}^{(2)\prime} = \frac{\dot{S}_{ij}^{(2)\prime}}{2\mu_2} + \dot{\lambda} S_{ij}^{(2)\prime} \quad \text{for} \quad I_2^{(2)\prime} = Y_2^2. \end{gathered} \tag{3}$$

Here $I_2^{(2)\prime}$ is the second invariant of the actual stress deviator tensor, Y_2 is the yield strength of the matrix material, the dot denotes the derivative with respect to time, and the parameter $\dot{\lambda}$ is found from the condition $I_2^{(2)\prime} = Y_2^2$.

Averaging system (2) over the inclusion volume and system (3) over the matrix volume, we obtain the following equations, which relate the mean stresses and strains in the matrix:

$$\begin{gathered} \varepsilon_{kk}^{(2)} = -\frac{p^{(2)}}{K_2}, \quad e_{ij}^{(2)} = \frac{S_{ij}^{(2)}}{2\mu_2} \quad \text{for} \quad I_2^{(2)} < Y_2^2, \\ \dot{e}_{ij}^{(2)} = \frac{\dot{S}_{ij}^{(2)}}{2\mu_2} + \dot{\lambda} S_{ij}^{(2)} \quad \text{for} \quad I_2^{(2)} = Y_2^2, \quad I_2^{(2)} = \frac{1}{V_2} \int_{V_2} I_2^{(2)\prime} \, dV \end{gathered} \tag{4}$$

and in the inclusion:

$$\varepsilon_{kk}^{(1)} = -\frac{p^{(1)}}{K_1}, \quad e_{ij}^{(1)} = \frac{S_{ij}^{(1)}}{2\mu_1}, \tag{5}$$

where $p^{(1)}$ and $S_{ij}^{(1)}$ are the mean pressure and stress tensor deviator in the inclusion, and $p^{(2)}$ and $S_{ij}^{(2)}$ are the corresponding quantities in the matrix. Let a uniform stress σ_{ij} be applied to the external boundary of the cell. We choose a Cartesian coordinate system (x_1, x_2, x_3) whose origin is in the center of the cell (Fig. 1) and whose axes coincide with the main axes of the stress tensor σ_{ij}. In this case, we have $\sigma_{ij} = \sigma_i \delta_{ij}$ and $\varepsilon_{ij} = \varepsilon_i \delta_{ij}$ (no summation over i is performed). Substituting these relations into system (1), we have

$$\varepsilon_i = m_1 \varepsilon_i^{(1)} + m_2 \varepsilon_i^{(2)}, \quad p = m_1 p^{(1)} + m_2 p^{(2)}, \quad S_i = m_1 S_i^{(1)} + m_2 S_i^{(2)}. \tag{6}$$

If we substitute $\varepsilon_i^{(1)}$ and $\varepsilon_i^{(2)}$ from Eqs. (4) and (5) into the right side of the first equation in (6), we obtain the dependence $\varepsilon_i = \tilde{\varphi}_i(p^{(1)}, p^{(2)}, S_i^{(1)}, S_i^{(2)})$. With account of the last two equations in (6), the number of independent variables can be reduced to two, for example, $\varepsilon_i = \varphi_i(p^{(1)}, S_i^{(1)})$. Thus, the problem is to express $p^{(1)}$ and $S_i^{(1)}$ in terms of p and S_i, where $\sigma_i = -p + S_i$ is the stress applied to the cell. We have to solve the problem on the cell (Fig. 1), i.e., the equilibrium equations with the boundary conditions

$$\frac{\partial \sigma_{ij}'}{\partial x_i} = 0, \qquad \sigma_{ij}'\Big|_{r=b} = \sigma_{ij}, \qquad u_i'\Big|_{r=0} = 0,$$

which should be supplemented by closing relations (2) and (3). In the elastoplastic case, there is no exact solution to this problem; therefore, an approximate solution is constructed below.

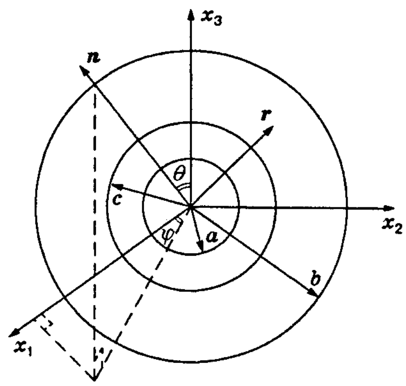

Fig. 1

First, we consider a simpler problem, where a pressure p is applied to the external boundary of the cell. The solution of this problem is known [7]; therefore, we formulate the problem and show the solution. Along with the Cartesian coordinate system, we introduce the spherical coordinates $(r,\ \theta,\ \varphi)$ (Fig. 1), in which the equilibrium equation and the boundary conditions have the form

$$\frac{d\sigma'_r}{dr} + \frac{2(\sigma'_r - \sigma'_\varphi)}{r} = 0, \qquad \sigma'_\theta = \sigma'_\varphi, \qquad u'_\theta = u'_\varphi, \qquad \sigma'_r\Big|_{r=b} = -p.$$

The system of equations should be supplemented by the conditions of continuity of normal stresses and displacements at the matrix–inclusion interface and zero displacements at the point $r = 0$:

$$u'\Big|_{a-0} = u'\Big|_{a+0}, \qquad \sigma'_r\Big|_{a-0} = \sigma'_r\Big|_{a+0}, \qquad u'\Big|_{r=0} = 0.$$

Here $u' = u'_r(r)$ is the displacement of the material along r. Following [7], we write the solution of this problem. In the region occupied by the inclusion, the strain is uniform and is determined from the formula

$$\varepsilon^{(1)}_{kk} = -\frac{p^{(1)}}{K_1}, \qquad u' = -\frac{p^{(1)}r}{3K_1}. \tag{7}$$

The stresses in the material in the elastic case $|p| < |p_0|$ are given by the formulas

$$\begin{gathered} \sigma'_r = -p + \Delta p\Big(1 - \frac{b^3}{r^3}\Big)\Big/\Big(1 - \frac{b^3}{a^3}\Big), \quad \Delta p = p - p^{(1)}, \\ \sigma'_\varphi = -p + \Delta p\Big(1 + \frac{b^3}{2r^3}\Big)\Big/\Big(1 - \frac{b^3}{a^3}\Big), \quad \sigma'_\theta = \sigma'_\varphi. \end{gathered} \tag{8}$$

In the elastoplastic case $|p_0| < |p| < |p_*|$, the stresses in the matrix have the form

$$\sigma'_r = -p^{(1)} + 2\text{æ}Y_2 \ln\frac{r}{a}, \quad \sigma'_\varphi = -p^{(1)} + \text{æ}Y_2 + 2\text{æ}Y_2 \ln\frac{r}{a}, \quad \sigma'_\theta = \sigma'_\varphi, \quad \text{æ} = \begin{cases} 1, & p < 0, \\ -1, & p > 0 \end{cases}$$

for $a < r \leqslant c$ and (9)

$$\sigma'_r = -p + \frac{2}{3}\text{æ}Y_2\Big(\frac{c}{b}\Big)^3\Big(1 - \Big(\frac{b}{r}\Big)^3\Big), \quad \sigma'_\varphi = -p + \frac{2}{3}\text{æ}Y_2\Big(\frac{c}{b}\Big)^3\Big(1 + \frac{b^3}{2r^3}\Big), \quad \sigma'_\theta = \sigma'_\varphi,$$

$$p_0 - p_0^{(1)} = -\frac{2}{3}\text{æ}Y_2\Big(1 - \Big(\frac{a}{b}\Big)^3\Big), \quad p_* - p_*^{(1)} = -\frac{2}{3}\text{æ}Y_2 \ln\Big(\frac{b}{a}\Big)^3$$

for $c < r \leqslant b$. Here c is the radius of the plastic zone (Fig. 1), p_0 is the pressure at which the plastic zone appears, p_* is the pressure at which the plastic zone fills the entire volume of the matrix, and the superscript (1) indicates the corresponding pressures in the inclusion, which are found below from the condition of displacement continuity at $r = a$. Stresses (8) and (9) satisfy the continuity condition for σ'_r at the interface between the matrix and inclusion. The displacement of the matrix material for $r > a$ is given by the formulas

$$u = -\frac{rp^{(2)}}{3K_2} + \delta u, \quad \delta u = \begin{cases} -(p - p^{(1)})\dfrac{b^3a^3}{4\mu_2(b^3 - a^3)r^2}, & |p| < |p_0|, \\ \dfrac{æY_2}{6\mu_2}\dfrac{c^3}{r^2}, & |p_0| \leqslant |p| < |p_*|, \end{cases} \tag{10}$$

$$p - p^{(1)} + \frac{2}{3}æY_2\Big(1 - \Big(\frac{c}{b}\Big)^3 + 3\ln\frac{c}{a}\Big) = 0.$$

The radius of the plastic zone c is determined from the last equation in (10). We use the condition $u\big|_{a-0} = u\big|_{a+0}$ to determine $p^{(1)}$. First, we consider an elastic case without a plastic zone. We find $p^{(1)}$ from the condition of displacement continuity at $r = a$. Assuming $r = a$ in formulas (7) and (10) and equating them to an accuracy of the terms $O(m_1)$, we obtain

$$\frac{p^{(1)}}{K_1} = \frac{p}{K_2} + \frac{(1+\nu_2)(p - p^{(1)})}{2(1 - 2\nu_2)K_2}. \tag{11}$$

We take into account that

$$\frac{1}{\mu_2} = \frac{2}{3}\frac{1+\nu_2}{1-2\nu_2}\frac{1}{K_2}, \tag{12}$$

where ν_2 is the Poisson's ratio of the matrix. Resolving Eq. (11) relative to $p^{(1)}$, we obtain

$$p^{(1)} = p\Big(1 - (K_2 - K_1)\Big/\Big(K_2 + K_1\frac{1+\nu_2}{2(1-2\nu_2)}\Big)\Big). \tag{13}$$

This solution allows us to find the relationship between ε_{kk} and p. Substituting (4) and (7) into the first equation of (6), we find

$$\varepsilon_i = -m_2\frac{p^{(2)}}{3K_2} - m_1\frac{p^{(1)}}{3K_1},$$

from which, with account of (6) and (13), we have

$$\varepsilon_{kk} = -\frac{p}{K}, \quad K = K_2\Big(1 + \frac{3m_1(1-\nu_2)(K_1 - K_2)}{(1+\nu_2)K_1 + 2(1-2\nu_2)K_2}\Big). \tag{14}$$

Note that formula (14) for K coincides with the corresponding formula (4.7) in [1]. In the elastoplastic case $|p| > |p_0|$, a plastic zone $r = c$ appears in the vicinity of the inclusion; the displacements in the matrix are determined by formulas (10) at $|p_0| \leqslant |p| < |p_*|$. Equating the displacements in the matrix (10) and in the inclusion (7) at $r = a$, we obtain the following equation for $p^{(1)}$:

$$\frac{p^{(1)}}{K_1} = \frac{p}{K_2} - \frac{æ}{3K_2}\frac{1+\nu_2}{1-2\nu_2}Y_2\frac{m_p}{m_1}. \tag{15}$$

Here $m_p = (c/b)^3$ is the volume fraction occupied by the plastic zone. The radius of the plastic zone c is determined from the last equation of system (10). The solution of this equation can be found numerically and, as shown in [5, 6], approximated by the Gurson formula

$$\Big(\frac{c}{b}\Big)^3 = m_p = 1 - m_e, \quad m_e \approx \sqrt{\frac{1+m_1^2}{m_2} - \frac{2m_1}{m_2}\cosh\frac{3(p - p^{(1)})}{2Y_2}}, \tag{16}$$

where m_e is the volume fraction of the cell in an elastic state. According to [5, 6], m_p is a monotonic (increasing) function of $p - p^{(1)}$; therefore, Eq. (15) for a given p ($|p| > |p_0|$) has the unique root $p^{(1)}$, which is found below for some specific cases using the numerical Newton method. Substituting (15) into (13), we find the relationship between ε_{kk} and p in the elastoplastic case

$$\varepsilon_{kk} = -\frac{p}{K_2} - \frac{m_1}{m_2}\frac{p - p^{(1)}}{K_2} + \frac{æY_2}{3K_2}\frac{1+\nu_2}{1-2\nu_2}m_p, \quad m_p = \left(\frac{c}{b}\right)^3, \quad |p_0| < |p| \leqslant |p_*|. \tag{17}$$

Here $p^{(1)}$ is found from Eq. (15) and m_p from Eq. (16). If we write Eq. (17) in the form $\varepsilon_{kk} = -p/K$, it follows that $K = K(p, m_1)$. For $|p| > |p_*|$, the averaged "modulus" of volume compression remains constant and equal to $K = K(p_*, m_1)$. The pressure p_0 at which the plastic zone appears is determined from the condition $m_p(p_0) = m_1$. Substituting this condition into Eq. (15), we rewrite it together with the second to the last equation of system (9) in the form

$$p_0 - p_0^{(1)} = -\frac{2}{3}æY_2 m_2, \quad \frac{K_2}{K_1}p_0^{(1)} = p_0 - \frac{1}{3}æY_2\frac{1+\nu_2}{1-2\nu_2}.$$

The solution of this system of equations is

$$p_0 = -\frac{2}{3}æY_2\Big(m_2 + \frac{K_1}{2K_2}\frac{1+\nu_2}{1-2\nu_2}\frac{K_2}{K_2-K_1}\Big). \tag{18}$$

Considering successively compression $æ = -1$ ($p_0 > 0$) and expansion $æ = 1$ ($p_0 < 0$) of the heterogeneous material, we find that Eq. (18) has a solution only in the case $K_2 > K_1$. This does not mean, however, that the plastic zone can arise only in the case where the volume compression coefficient of the matrix K_2 is greater than K_1 in the inclusion. According to [7], the components of the stress tensor (9) in the yield state $a < r < c$ satisfy the condition

$$\sigma'_\varphi - \sigma'_r = æY_2, \quad æ = \begin{cases} 1, & p < 0, \\ -1, & p > 0. \end{cases} \tag{19}$$

The sign of $æ$ in Eq. (19) is chosen from the solution of the elastic problem, for which from (8) we obtain

$$\sigma'_\varphi - \sigma'_r = -\frac{3}{2}\frac{m_1}{m_2}(p - p^{(1)})\frac{b^3}{r^3}. \tag{20}$$

According to Eq. (13), we have

$$p - p^{(1)} = p(K_2 - K_1)\Big/\Big(K_2 + K_1\frac{1+\nu_2}{2(1-2\nu_2)}\Big), \tag{21}$$

therefore, for $K_2 > K_1$, it follows from (20) and (21) that $\sigma'_\varphi - \sigma'_r \sim -p$, wherefrom we obtain

$$\sigma'_\varphi - \sigma'_r > 0 \quad \text{for} \quad p < 0, \qquad \sigma'_\varphi - \sigma'_r < 0 \quad \text{for } p > 0. \tag{22}$$

Inequalities (22) serve as a basis for choosing the sign of $æ$ in Eq. (19). For $K_2 < K_1$, it follows from (20) and (21) that $\sigma'_\varphi - \sigma'_r \sim p$; hence, the following inequalities are valid:

$$\sigma'_\varphi - \sigma'_r < 0 \quad \text{for } p < 0, \qquad \sigma'_\varphi - \sigma'_r > 0 \quad \text{for } p > 0. \tag{23}$$

From inequalities (23), we find that, for $K_2 < K_1$, the signs of $æ$ in the yield condition (19) should be reversed:

$$\sigma'_\varphi - \sigma'_r = æY_2, \quad æ = \begin{cases} -1, & p < 0, \\ 1, & p > 0. \end{cases} \tag{24}$$

Thus, Eqs. (19) and (24) allow us to determine the sign of $æ$ in (18) for arbitrary K_1 and K_2. Hence, a plastic zone will appear for all ratios K_2/K_1 other than unity.

We consider the more general case where a uniform stress $\sigma_{ij} = (-p + S_i)\delta_{ij}$ is applied to the cell boundary $r = b$ (no summation over i is performed). As noted above, there is no exact solution to this problem. Therefore, following [4–6], we construct an approximate solution, which satisfies the boundary conditions

$$\sigma'_r\big|_{r=b} = -p + S_1 n_1^2 + S_2 n_2^2 + S_3 n_3^2, \tag{25}$$

where $n_1 = \sin\theta\cos\varphi$, $n_2 = \sin\theta\sin\varphi$, and $n_3 = \cos\theta$ are the components of the normal vector to the cell surface (Fig. 1). Approximate solutions of the equilibrium equations with the boundary condition (25) are determined by formulas (7)–(10) with the following substitution: $-p \to \sigma_n = -p + S_1 n_1^2 + S_2 n_2^2 + S_3 n_3^2$. The displacement in the Cartesian coordinate system at the point with the coordinates (r, θ, φ) is determined by the formula $u_i = n_i u(r, n_1, n_2, n_3)$; hence, the displacement of the points formed by intersection of the Cartesian axes with a sphere of radius r is determined by formulas (7) and (10) with the substitution $-p \to -p + S_i$. As a result, the displacements in the inclusion $r < a$ are determined by the formula

$$u_i = -p^{(1)} r/(3K_1) + S_i^{(1)} r/(2\mu_1), \tag{26}$$

and the displacements in the matrix $a \leqslant r < b$ are found from the formula

$$u_i = -p^{(2)} r/(3K_2) + S_i^{(2)} r/(2\mu_2) + \delta u_i. \tag{27}$$

In the elastic case $|p| < |p_0|$ we have

$$\delta u_i = -\frac{p - p^{(1)}}{6K_2} \frac{1+\nu_2}{1-2\nu_2} \frac{m_1}{m_2} \frac{b^3}{r^2} + \frac{S_i - S_i^{(1)}}{4\mu_2} \frac{m_1}{m_2} \frac{b^3}{r^2}. \tag{28}$$

In the plastic case $|p_0| \leqslant |p| < |p_*|$, we obtain

$$\delta u_i = -\frac{æY_2}{6\mu_2} \frac{c_i^3}{r^2}, \tag{29}$$

where c_i is determined from the equation

$$p - p^{(1)} - \left(S_i - S_i^{(1)}\right) + \frac{2}{3} æY_2 \left(1 - \left(\frac{c_i}{b}\right)^3 + 3 \ln \frac{c_i}{a}\right) = 0. \tag{30}$$

Expanding expression (30) into a power series $(S_i - S_i^{(1)})/Y_2$ and substituting into Eq. (29), we find

$$\delta u_i = \frac{æY_2}{9K_2} \frac{1+\nu_2}{1-2\nu_2} \frac{b^3}{r^2} m_p + \frac{S_i - S_i^{(1)}}{4\mu_2} \frac{m_p}{m_e} \frac{b^3}{r^2}. \tag{31}$$

As shown below, the components of the stress deviator satisfy the inequality $(3/2)(S_1^2 + S_2^2 + S_3^2) \leqslant Y^2$, from which we obtain $|S_i| \leqslant \sqrt{2/3} Y$ for an individual component. The averaged yield strength Y is determined from formula (39) derived below, from which we obtain the inequality $Y < Y_2$. Thus, we have a limitation from above

$$\frac{|S_i - S_i^{(1)}|}{Y_2} < \frac{|S_i|}{Y_2} = \frac{|S_i|}{Y} \frac{Y}{Y_2} \leqslant \sqrt{\frac{2}{3}} \frac{Y}{Y_2} < 1,$$

which justifies expansion (31).

From the displacement continuity condition (26), (27) at $r = a$, taking into account Eqs. (6), (11), (12), (15), (28), and (31), we obtain the equations

$$\frac{S_i^{(1)}}{2\mu_1} = \frac{S_i}{2\mu_2} + \frac{S_i - S_i^{(1)}}{4\mu_2}, \quad |p| < |p_0|, \qquad \frac{S_i^{(1)}}{2\mu_1} = \frac{S_i}{2\mu_2} + \frac{S_i - S_i^{(1)}}{4\mu_2 m_1} \frac{m_p}{m_e}, \quad |p_0| \leqslant |p| < |p_*|.$$

Resolving these equations, we obtain the following formulas for $S_i^{(1)}$:

$$\begin{aligned} S_i - S_i^{(1)} &= S_i(\mu_2 - \mu_1) \Big/ \left(\mu_2 + \frac{1}{2}\mu_1\right), \quad |p| < |p_0|, \\ S_i - S_i^{(1)} &= S_i(\mu_2 - \mu_1) \Big/ \left(\mu_2 + \frac{m_p}{2m_1 m_e}\mu_1\right), \quad |p_0| \leqslant |p| < |p_*|. \end{aligned} \tag{32}$$

Substituting (4) into the first equation of system (6) and separating the spherical and deviator components of the strain tensor, we obtain

$$e_i = m_2 S_i^{(2)}/(2\mu_2) + m_1 S_i^{(1)}/(2\mu_1), \tag{33}$$

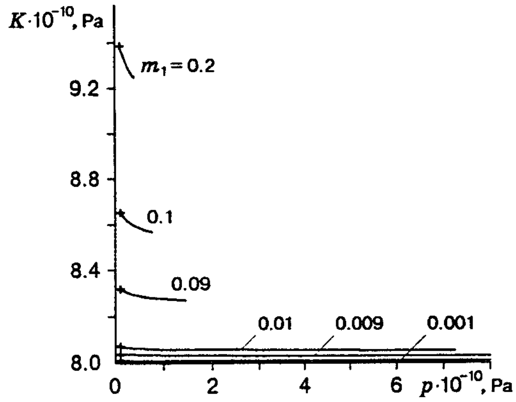

Fig. 2

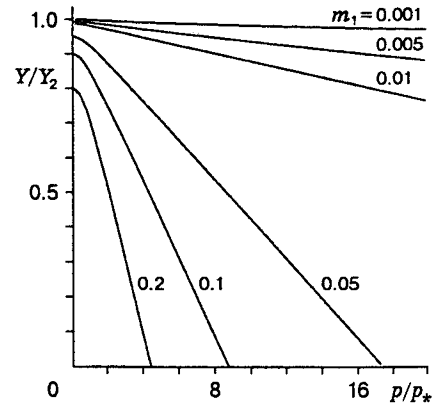

Fig. 3

where $S_i^{(2)} = (S_i - m_1 S_i^{(1)})/m_2$. Using formulas (32), we can rewrite Eq. (33) in the form

$$e_i = \frac{S_i}{2\mu}, \qquad \mu = \begin{cases} \mu_2\Big(1 + \dfrac{3m_1(\mu_1 - \mu_2)}{\mu_1 + 2\mu_2}\Big), & |p| < |p_0|, \\ \mu_2\Big(1 + \dfrac{m_1(\mu_1 - \mu_2)(m_p m_2 + 2m_1 m_e)}{m_p m_2 \mu_1 + 2m_1 m_e \mu_2}\Big), & |p_0| \leqslant |p| < |p_*|. \end{cases} \tag{34}$$

Formulas (14), (17), and (34) are the closing relations for a heterogeneous medium consisting of a matrix and spherical inclusions. It follows from these formulas that, when the effect of the plastic zone is taken into account, the averaged "moduli" of shear μ and volume compression K depend not only on porosity m_1, but also on the pressure p.

Until now we assumed that the entire material of the matrix or a part of it is in the elastic state $I_2^{(2)} < Y_2^2$. If the entire material of the matrix transforms to the plastic state $I_2^{(2)} = Y_2^2$, its plastic flow is observed. According to (4), the increment of the strain deviator is

$$de_{ij}^{(2)} = \frac{dS_{ij}^{(2)}}{2\mu_2} + d\lambda\, S_{ij}^{(2)}, \tag{35}$$

where $d\lambda$ is found from the condition

$$\frac{1}{V_2}\int\limits_{V_2} \frac{3}{2} S_{ij}^{(2)\prime} S_{ij}^{(2)\prime}\, dV = Y_2^2. \tag{36}$$

Substituting the deviator of microstresses $S_{ij}^{(2)\prime}$ as the sum of mean $S_{ij}^{(2)} = \dfrac{1}{V_2}\displaystyle\int\limits_{V_2} S_{ij}^{(2)\prime}\, dV$ and fluctuating $\tilde{S}_{ij}^{(2)}$ stresses $S_{ij}^{(2)\prime} = S_{ij}^{(2)} + \tilde{S}_{ij}^{(2)}$, we obtain the following expression from Eq. (36):

$$\frac{3}{2} S_{ij}^{(2)} S_{ij}^{(2)} dV + \frac{1}{V_2}\int\limits_{V_2} \frac{3}{2} \tilde{S}_{ij}^{(2)} \tilde{S}_{ij}^{(2)}\, dV = Y_2^2. \tag{37}$$

Here we used $\displaystyle\int\limits_{V_2} \tilde{S}_{ij}^{(2)}\, dV = 0$. By analogy with [4–6], we assume that $\tilde{S}_{ij}^{(2)}$ arise under the action of the pressure p applied to the cell surface. In this case, we have

$$\frac{1}{V_2}\int\limits_{V_2} \frac{3}{2} \tilde{S}_{ij}^{(2)} \tilde{S}_{ij}^{(2)} dV = \frac{1}{V_2}\int\limits_{V_2} \frac{3}{2}\Big((\tilde{S}_r^{(2)})^2 + 2(\tilde{S}_\varphi^{(2)})^2\Big)\, dV, \tag{38}$$

where $\tilde{S}_r^{(2)} = \sigma_r' + p'$, $\tilde{S}_\varphi^{(2)} = \sigma_\varphi' + p'$, $p' = -(1/3)(\sigma_r' + 2\sigma_\varphi')$, and σ_r' and σ_φ' are the microstresses in the cell induced by the pressure p, which are calculated by formulas (8) and (9). Substituting σ'_r and σ'_φ from (8) and (9) into (37) and (38), we obtain

$$\frac{3}{2}S_{ij}^{(2)}S_{ij}^{(2)} = Y^2, \tag{39}$$

$$Y^2 = \begin{cases} Y_2^2 m_2^2 - (9/4)(p - p^{(1)})^2 m_1, & |p| < |p_0|, \\ Y_2^2 m_2 m_e^2, & |p_0| < |p|, \end{cases} \tag{40}$$

where $p^{(1)}$ and m_e are determined above. Using formula (16), we approximate expression (40) by the Gurson formula

$$Y^2 = Y_2^2\Big(1 + m_1^2 - 2\,m_1 \cosh\Big(\frac{3}{2}\,\frac{p - p^{(1)}}{Y_2}\Big)\Big).$$

Equations (35), (39), and (40) allow us to find the increments of the strain tensor for a plastic flow of the matrix material. Figures 2 and 3 show the averaged "modulus" K and the yield strength Y as functions of the pressure p for different values of m_1 calculated using the above-derived formulas (14), (15), (17), and (39). The radius of the plastic zone was found from the last equation of system (10), which was solved numerically using the Newton method with account of (15). It was assumed in the calculations that the cell consists of an aluminum matrix ($K_2 = 8 \cdot 10^{10}$ Pa, $\mu_2 = 2.48 \cdot 10^{10}$ Pa, and $Y_2 = 3 \cdot 10^8$ Pa) and a carbide SiC inclusion ($K_1 = 2.13 \cdot 10^{11}$ Pa and $\mu_1 = 1.87 \cdot 10^{11}$ Pa). The crosses in Fig. 2 indicate the points $p = p_0$ where the plastic zone originates. The dependence $K(p)$ in Fig. 2 was constructed until $p = p_*$; for $p > p_*$ the averaged "modulus" K remains constant and equal to $K(p_*, m_1)$. The magnitude of K decreases by 10–20% under the influence of the plastic zone, and the smaller m_1, the weaker its effect on K. Note that the dependence of the averaged shear "modulus" μ on pressure has a similar character. For $m_1 < 10^{-3}$, the values of K, μ, and Y are almost constant and equal to K_2, μ_2, and Y_2, respectively. As the pressure $|p|$ increases, the yield strength Y decreases (Fig. 3) and vanishes when the pressure reaches the critical value $|p_*|$. As a result, the mean stress tensor in the composite σ_{ij} becomes spherical. Thus, for $|p| > |p_*|$, the mechanical properties of a composite become similar to the properties of a fluid.

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